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An Investigation into the proper Method of Determining the Amount of an Annuity forborn and improved at Interest, during the existence of a given Life. By PETER HARDY, F.R.S., one of the Vice-Presidents of the Institute of Actuaries.

[Read before the Institute, 26th January, 1857, and ordered by the Council to be printed.]

ART. 1. THE late Mr. William Morgan, in the second edition of his *Principles and Doctrine of Assurances and Annuities on Lives*, published in the year 1821, has, in the 7th Problem of that work, investigated "the method of determining the amount of an annuity forborn and improved at compound interest during the continuance of a given life." This problem he has subsequently discussed in an algebraical form, in Note V., page 154; and he has moreover given a table (Table VII.) showing the sum to which an annuity forborn and improved at 4 per cent. compound interest will amount, on the extinction of a given life, computed according to the foregoing method, from the Northampton Table of Observations.

ART. 2. The late Mr. Griffith Davies, in his *Tables of Life Contingencies*, published in the year 1825, has given (page 38) a rule for summing an annuity on a given life, when forborn and improved at compound interest; and in his larger but incomplete work, printed during his lifetime, but only to a certain extent completed and published since his death, has entered (page 237) very fully into an arithmetical investigation of this particular problem, because, as he subsequently observes (page 245), "this

mode of illustration has been given to enable such readers as may be unacquainted with mathematical investigations to satisfy themselves as to the accuracy of the method here given for finding the amount of a life annuity forborn; which method differs essentially, not only in form, but also in the results produced, from that given by Mr. Morgan in the 7th Problem of the last edition of his *Work on Assurances, &c.*"

ART. 3. A careful examination of the two methods here alluded to will lead the scientific reader to suspect that these two eminent writers have been, in point of fact, investigating two wholly different questions.

ART. 4. Mr. Morgan's method of investigating the problem is as follows. He says:—"If the life exists one year, the accumulation will be just £1. If it exists two years, the accumulation will be £1, increased by its interest for a year. If it exists three years, the accumulation will be £1, increased by its interest for two years, and so on for the other years;* so that the whole accumulation will be expressed by the sum of the series arising from the multiplication of these several amounts of £1 into the respective probabilities that the life exists one, two, three, &c., years, &c. &c.;" and, further (in Note V., page 154), he gives the series representing the amount of the annuity, so forborn during the life of A, as

$$\frac{a_1}{a} + \frac{a_2 r}{a} + \frac{a_3 r^2}{a} + \frac{a_4 r^3}{a} + \&c. \dots$$

where r = the amount of £1 increased by its interest for one year, and $a, a_1, a_2, a_3, \&c.$, respectively represent the numbers living in any table of observations at the ages of A, A + 1, A + 2, A + 3, &c.

ART. 5. Mr. Davies' solution of the problem under consideration differs, as it will presently appear, very materially from the above. Having previously computed what may be termed an auxiliary table, by the columnar method, consisting of columns D, N, and M, according to any given rate of interest and on any given table of observations, then $\frac{N}{M}$, at the given age, will, by Mr. Davies' formula, represent the amount of an annuity forborn during the continuance of the life.

ART. 6. In order to determine whether there is any, and if any, what, sort of resemblance between these two solutions, it will

* This is not quite clearly expressed: instead of the word 'accumulation' merely being employed, Mr. Morgan should have said, 'the accumulation on the first payment at the end of the first year, and on the second payment, &c., &c.; its meaning is, however, sufficiently obvious.

be first necessary to throw $\frac{N}{M}$ into an algebraical form, employing the same symbols as those used in the 4th Art.

ART. 7. Mr. Davies has given the following rule for the construction of the Columns D, N, and M, viz.—for Column D, the *numbers living* in the table at each age are to be respectively multiplied into £1, discounted for as many years as are *equal to the given age*; Column N is then formed by summing the successive values in Column D, at all *older ages* than the given age, for a new value in Column N at the given age; and to construct Column M the *numbers dying* are to be respectively multiplied into £1, discounted for as many years as are equal to each age *increased by unity*, and the sum of these successive values *at the given age* and all older ages will constitute the value in Column M at the given age; therefore, at age A, the following will be the algebraical arrangement of the three Columns D, N, and M, according to any table of observations:—

Ages.	Numbers Living.	D.	N.	M.
A	a	$\frac{a}{r^A}$	$\frac{a_1}{r^{A+1}} + \frac{a_2}{r^{A+2}} + \frac{a_3}{r^{A+3}} + \&c.$	$\frac{a-a_1}{r^{A+1}} + \frac{a_1-a_2}{r^{A+2}} + \frac{a_2-a_3}{r^{A+3}} + \frac{a_3-a_4}{r^{A+4}}$
A+1	a_1	$\frac{a_1}{r^{A+1}}$		
A+2	a_2	$\frac{a_2}{r^{A+2}}$		
&c.	&c.	&c.		

Consequently, at age A,

$$\frac{N}{M} = \frac{\frac{a_1}{r^{A+1}} + \frac{a_2}{r^{A+2}} + \frac{a_3}{r^{A+3}} \&c.}{\frac{a-a_1}{r^{A+1}} + \frac{a_1-a_2}{r^{A+2}} + \frac{a_2-a_3}{r^{A+3}} \&c.}$$

Now by multiplying r^A into both numerator and denominator of the above fraction, we shall convert it into the following fraction, viz.—

$$\frac{\frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} \&c.}{\frac{a-a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} \&c.}$$

And again, by multiplying the numerator and denominator by the common quantity a^{-1} , we shall obtain

$$\frac{\frac{a_1}{ar} + \frac{a_2}{ar^2} + \frac{a_3}{ar^3} \&c.}{\frac{a-a_1}{ar} + \frac{a_1-a_2}{ar^2} + \frac{a_2-a_3}{ar^3} \&c.}$$

A very casual inspection of this last resulting fraction will render it at once manifest that the series representing the numerator is identical with the series which represents the present value of an annuity of £1 during the life of A, symbolized in my *Notation* (published sixteen or seventeen years ago) by $\overset{\circ}{IA}$; and further, that the series representing the denominator of the above fraction is, in like manner, identical with the series which represents the present value of a reversion of £1 payable on the extinction of the life of A, symbolized also in my *Notation* by $\overset{\circ}{AI\overset{\circ}{I}}$:

consequently, the formula $\frac{N}{M}$ becomes equal to $\frac{\overset{\circ}{IA}}{\overset{\circ}{AI\overset{\circ}{I}}}$; or in other

words, according to Mr. Davies' method, the *amount* of an annuity of £1 forborn during the life of A is equal to quotient which arises from the division of the *present value* of the annuity by the *present value* of the reversion at the given age. Now if p represent the *ordinary annual premium*, corresponding to the single premium ($\overset{\circ}{AI\overset{\circ}{I}}$), and if P represent the annual premium payable at the *end of each year*, instead of at the beginning of the year as

in ordinary cases,* then $\frac{\overset{\circ}{IA}}{\overset{\circ}{AI\overset{\circ}{I}}}$ becomes equal to $\frac{1}{P}$, or to the reci-

procal of an annual premium on the given life, payable at the end instead of at the beginning of each year; this may be very readily

shown to be the case, because $p = \frac{\overset{\circ}{AI\overset{\circ}{I}}}{\overset{\circ}{IA} + 1}$, and $P = \frac{\overset{\circ}{AI\overset{\circ}{I}}}{\overset{\circ}{IA}}$; con-

sequently, $\frac{\overset{\circ}{IA}}{\overset{\circ}{AI\overset{\circ}{I}}} = \frac{1}{P}$ Q. E. D.

ART. 8. The solution which we have thus finally reached does not fail to point out to us the principle on which Mr. Davies' rule is founded, viz.—that as P , annually improved at compound interest during the life of A, will produce or insure £1 on its ex-

* See a paper on this subject in the *Assurance Magazine*, vol. i.

tion, so £1 annually improved will in like manner, on the happening of the like event, produce or insure $\text{£}\frac{1}{P}$.

ART. 9. In this stage of the investigation it must have become sufficiently obvious that no sort of resemblance exists between the two methods of solution, unless under the particular conditions hereafter to be described; and it may be further shown that the arithmetical results, although computed from the same table of observations and at the same rate of interest, are as widely different as the rules themselves and as the reasoning by which those rules have been deduced.

ART. 10. The following short table will exhibit the arithmetical results of the problem at the under-mentioned ages, if solved by Mr. Morgan's method or by Mr. Davies' method.

Table showing the Amount of an Annuity of £1 forborn and improved during the continuance of Lives of the under-mentioned Ages, computed from the Northampton Table, at 4 per cent.

Age.	By Mr. Davies.	By Mr. Morgan.
1	30.34	137.79
10	60.95	137.87
• 20	46.49	94.50
40	29.07	44.88
60	14.72	18.50

ART. 11. These results exhibit differences far too serious in amount to be passed over without some inquiry. I think it was in the year 1839 that I drew the attention of Professor De Morgan to this startling difference in the arithmetical results obtained from two rules, both professing to solve the same problem; and in the *Companion to the British Almanack for the Year 1842*, page 13, Mr. De Morgan briefly notices this circumstance and gives a comparison of Davies' and Morgan's solutions—without, however, particularly examining Mr. Davies' method. The problem has been also solved by Mr. Benwell, and subsequently, in an indirect form, by Dr. Farr,* and both these two last methods of solution agree entirely with that of Mr. Davies; but, with the exception of Professor De Morgan, no one, to my knowledge, has taken any notice of the two methods as compared with one another.

ART. 12. It is not, however, a little singular, that in two extreme cases which this and other similar problems present, the two

* See *Twelfth Report of Registrar-General*, Table XV., page 10 in Appendix.

methods of solution become identical. For example: if, instead of being forborn and improved during the continuance of a given life, the annuity of £1 is to be forborn and improved *for a term of years certain*—or if the annuity of £1 be merely *forborn during the continuance of a life*, but without improvement at interest—in both these extreme instances the arithmetical results coincide, and the form of the solution as proposed by Mr. Morgan becomes identical with that proposed by Mr. Davies; it is, therefore, worth while to show this, in order to be better enabled to ascertain at what particular point and from what particular cause two methods of solution, which are identical at their extremities, diverge at all other points so widely from each other.

ART. 13. Mr. Morgan's solution has been already shown to be

$$\frac{a_1}{a} + \frac{a_2 r}{a} + \frac{a_3 r^2}{a} + \frac{a_4 r^3}{a} + \frac{a_5 r^4}{a} + \&c.$$

Now if the annuity be for a term certain instead of contingent for its duration on the existence of a life, the fractions $\frac{a_1}{a}, \frac{a_2}{a}, \frac{a_3}{a}, \&c.$, which respectively measure the probability of the life existing one, two, three, &c. years, each becomes equal to unity, and the series

$$\frac{a_1}{a} + \frac{a_2 r}{a} + \frac{a_3 r^2}{a} + \frac{a_4 r^3}{a} + \&c.$$

consequently becomes $1 + r + r^2 + r^3 + \&c. r^{n-1} = \frac{r^n - 1}{r - 1}$, equal also to $V_n \times r^n$ where V_n represents the present value of £1 per annum for n years certain.

ART. 14. Mr. Davies' solution has also been shown to be

$$\frac{\overset{\circ}{\text{IA}}}{\text{A}\overset{\circ}{\text{I}}1}.$$

Now if no life be involved in the annuity, or if, in other words, the annuity of £1 be forborn for n years certain, then $\overset{\circ}{\text{IA}}$, which represents the present value of an annuity on the life of A, becomes equal to V_n , or to the value of an annuity for n years merely, and $\text{A}\overset{\circ}{\text{I}}1$, which represents the value of £1 to be received on the extinction of the life of A, becomes equal to $\frac{1}{r^n}$, the present value of £1 to be received merely on the expiration of the term, or to the value of £1 discounted for n years: thus $\frac{\overset{\circ}{\text{IA}}}{\text{A}\overset{\circ}{\text{I}}1}$ becomes $\frac{V_n}{\frac{1}{r^n}} = V_n \times r^n$, as shown in the 13th Article. Thus proving that, in

the case of an annuity being forborn for a term of years certain, instead of during the continuance of a life, the two solutions are identical.

ART. 15. Let us now assume the other extreme case, and suppose the annuity to be forborn during the continuance of a given life, but to bear no interest whatever. In this case the successive terms r, r^2, r^3, r^4 , &c., become respectively equal to unity; because r , being equal to $(1+i)$ when i (the rate) has a positive value, becomes equal to $1+0$, or unity, when i has no value whatever; and the terms of the series given as Mr. Morgan's solution become, by the conversion of r and its powers into unity, equal to

$$\frac{a_1}{a} + \frac{a_2}{a} + \frac{a_3}{a} + \frac{a_4}{a} + \&c. = EA,$$

where EA represents the curtate expectation of the life of A . And as it has been repeatedly shown elsewhere that the value of an annuity on a given life, when interest is not involved in the question, is the same thing as the curtate expectation of the life—that is to say, IA becomes EA ,* and in like manner, $AI\overset{\circ}{1}$ becomes $A\overset{\circ}{E}1$, or unity†—so Mr. Davies' solution of $\frac{IA}{AI\overset{\circ}{1}}$ becomes equal

$\frac{EA}{A\overset{\circ}{E}1}$, or $\frac{EA}{1}$ equal to EA , as before. Again showing that, in the extreme case of money bearing no interest, the solutions are, as before, identical.

ART. 16. This last condition naturally leads us to the conviction that the curtate expectation of the life is the minimum of the amounts under consideration, or the lowest limit in the solution of the problem, whether by Mr. Morgan's method or by that of Mr. Davies—in other words, that if interest on money be allowed, the amount of an annuity forborn must exceed the expectation at any and every age at which the calculation may be made: and, further, that the higher the rate of interest, the greater will be the total amount; and, *vice versâ*, the lower the rate of interest the smaller will be the amount, but always in excess of the expectation.

* See *Assurance Magazine*, vol. ii., page 259.

† This may be thus shown:

$$AI\overset{\circ}{1} = \frac{IA+1}{r} - IA,$$

and, r being equal 1,

$$A\overset{\circ}{E}1 = \frac{EA+1}{1} - EA = 1.$$

Let us now apply this test to the two methods under consideration. By the Northampton Table, 4 per cent., the amount of an annuity forborn during the life of a child one year old—we learn from the little table in Art. 10—is, by Mr. Morgan's method, 137·79, while by Mr. Davies' method it is only 30·34; but the curtate expectation of the life of a child one year old is 32·24: consequently, if the assumption be true that the expectation is a minimum amount, then any rule which makes such amount, when bearing interest, less than the expectation, cannot be correct.

ART. 17. A very elegant proof of this may be found in an ingenious solution of the problem sent to me by my friend Mr. John Ware Stephenson, the Assistant Actuary of the Equitable, whose attention I had previously directed to this problem. This solution, while it amply confirms the correctness of Mr. Morgan's method, affords, in addition, an elegant proof of what has been advanced, and shown numerically in the foregoing Article. I have, for the sake of uniformity, changed one or two of Mr. Stephenson's symbols into such as we have been already in this paper accustomed to employ; but I give his analysis in his own form. Let $r=1+i$ and y equal the amount required, then, arranging the series by aid of Maclaurin's theorem, under the powers of i , we shall have from

$$\begin{aligned}
 y &= \frac{a_1}{a} + \frac{a_2 r}{a} + \frac{a_3 r^2}{a} + \dots + \frac{a_{n-1} r^{n-2}}{a} + \frac{a_n r^{n-1}}{a} \\
 ay &= a_1 + a_2 r + a_3 r^2 + \dots + a_{n-1} r^{n-2} + a_n r^{n-1} \\
 a \frac{dy}{di} &= (a_2 + 2 \cdot a_3 + 3a_4 r^2 + \dots + (n-2)a_{n-1} r^{n-3} + (n-1)a_n r^{n-2})i \\
 a \frac{d^2 y}{di^2} &= (2 \cdot a_3 + 2 \cdot 3a_4 r + \dots + (n-2)(n-3)a_{n-1} r^{n-4} + (n-1)(n-2)a_n r^{n-3}) \frac{i}{1 \cdot 2} \\
 a \frac{d^3 y}{di^3} &= \quad \&c. \quad \quad \quad \&c. \quad \quad \quad \&c. \\
 &\quad \quad \quad \therefore \\
 ay &= (a_1 + a_2 + a_3 + a_4 + \&c.) + (a_2 + 2a_3 + 3a_4, \&c.)i + (a_3 + 3a_4 + 6a_5 + 10a_6, \&c.)i^2 + \&c. \\
 y &= \frac{(a_1 + a_2 + a_3 + a_4, \&c.)}{a} + \frac{(a_2 + 2a_3 + 3a_4, \&c.)}{a}i + \frac{(a_3 + 3a_4 + 6a_5 + 10a_6, \&c.)}{a}i^2 + \&c.
 \end{aligned}$$

Now, if we suppose $i=0$, then all the terms in which i and its powers are involved, vanish. This, it will be seen, on the above supposition, is the case with all but the first term, which is, obviously, equal to the expectation of the life, or to the minimum value of the series; but when i has any positive value, which must always be the case when interest is allowed, then the amount of an annuity forborn must be greater than the expectation of life by all the terms

following the first—Q.E.D. By giving successive values to i and its powers, the value of the series may be readily numerically ascertained—inasmuch as, i being fractional, its powers rapidly diminish in value, and the whole series converges with extreme swiftness.

ART. 18. Taking a numerical example from the Northampton Table, 4 per cent., say at age 87 :

Age.	Living.	a .	Σa .	$^2\Sigma a$.	$^3\Sigma a$.	&c.
87	—	$a = 111$				
88	—	$a_1 = 83$	— 279			
89	—	$a_2 = 62$	— 196	— 522		
90	—	$a_3 = 46$	— 134	— 326	— 699	
91	—	$a_4 = 34$	— 88	— 192	— 373	— 666
92	—	$a_5 = 24$	— 54	— 104	— 181	⋮
93	—	$a_6 = 16$	— 30	— 50	— 77	⋮
94	—	$a_7 = 9$	— 14	— 20	— 27	⋮
95	—	$a_8 = 4$	— 5	— 6	— 7	⋮
96	—	$a_9 = 1$	— 1	— 1	— 1	⋮
						&c.

$$y = \frac{279}{111} + \frac{522}{111} (.04) + \frac{699}{111} (.04)^2 + \frac{666}{111} (.04)^3, \text{ \&c.}$$

$$y = \frac{279}{111} + \frac{20.88}{111} + \frac{1.118}{111} + \frac{.0426}{111}$$

$$y = \frac{301.041}{111} = 2.712 = \text{Value at age 87 by Morgan's Table VII.}$$

Now the first term is obviously equal to the curtate expectation of the life—that is, $\frac{279}{111} = 2.51$; and $2.51 + .5 = 3.01 =$ the tabular expectation of life (*see* Morgan, Table II.); and all the terms following the first must have a positive value, however small, if i have any positive value whatever. Therefore the amount of an annuity foreborn at any rate of interest, however low that rate may be, must exceed the curtate expectation of the life.

ART. 19. It now remains to inquire, whence the singular agreement of the two methods of solution in the two extreme cases noticed as above, viz.—when no life whatever is concerned in the annuity, and when no interest whatever is allowed on its forbearance? and why, also, the two methods should so entirely disagree, and differ in form and results, when the two elements, interest and vitality, are both present together?

ART. 20. On a careful reflection over and examination of this curious problem, there will, I think, be found to be considerable ambiguity in the very terms of the question. It will be therefore very desirable that we should investigate the whole subject somewhat carefully. In all financial questions which relate to life con-

tingencies there are necessarily present two elements, viz.—the interest of money, and the probabilities of life. All calculations which involve the former only are in themselves, singly and collectively, true, perfect, and complete, and the computed results will one and all be realized: but calculations in which the probabilities of life are involved must necessarily be imperfect as regards individual cases, some of which must be in excess, while others are in defect; such calculations are therefore only true in the mass, or in individual cases on the average. For example: suppose 100 annuities to be granted for different terms of years, some long and some short, each individual value will exactly provide each individual annuity, and all the aggregate values will provide all the aggregate annuities. But not so if 100 like annuities be granted on as many different lives of one common age, some of which will live a long term and some only a short term; for while the sum of all the values in the mass, or, what is the same thing, the sum of all the average values, will well and truly provide all the aggregate annuities, each separate or average value will completely fail to provide exactly (neither in excess nor in defect) its own separate annuity. This will be more clearly shown as we proceed in the inquiry.

ART. 21. Average, therefore, constitutes the foundation or basis of all calculations in which the probabilities of life are involved. The present value of an annuity, at any given age at which there are (*a*) persons living, in a table of observations, is only a quotient of the total amount of a present fund in hand, which, being improved at interest from year to year, will exactly provide the payment of £1 per annum to each and all of these *a* persons who respectively survive to the commencement of each subsequent year; and such a fund, when divided by the number of original subscribers (*a*), will yield a quotient which is an average sum or value belonging to each; and although the fund in the aggregate will, as I have before stated, provide all the annuities required, yet the individual share of any member, taken by itself, cannot be expected to do so exactly, neither more nor less—inasmuch as some of the subscribers will never live to receive one single payment of the annuity, while others will live to receive amounts far greater than their individual or average shares would possibly produce. Again: the present value of an assurance or reversion of £1 is, in like manner, a quotient of that present fund in hand, which, being improved at interest from year to year, will exactly provide a payment of £1 to each of *a* persons dying from year to year; but the average value or individual share of each cannot possibly be expected

to provide for its own payment, neither more nor less. In the two instances above given we have been treating of an average of an *already existing fund in hand*, but an average may with equal propriety be taken of a fund hereafter to be created or produced by forbearance and improvement. This, although sufficiently obvious, shall presently be more exactly shown: and I imagine that it can be made quite apparent, that while Mr. Davies' problem treats of the average of present values, Mr. Morgan's problem treats of the average of amounts not now in hand, but to be hereafter created by the outstanding of each individual payment and its improvement; or, more strictly speaking, the average of what each individual payment had amounted to on the extinction of each life.

ART. 22. If each individual subscriber of £1 per annum, to a fund to be improved at interest, be entitled to withdraw at his death the exact amount of his past payments and their improvement only, it is obvious that a succession of sums would be withdrawn from the common fund, increasing in amount each year: those who died in the first year of the contract getting nothing—as they would in fact have contributed nothing; and those who died in the n th year of the contract getting in all £($n-1$) and their successive improvement during ($n-2$) years.

ART. 23. This may be shown algebraically. The numbers living in the table being $a, a_1, a_2, a_3, a_4, a_5$, &c., it follows that the numbers dying in each year will be respectively $a-a_1, a_1-a_2, a_2-a_3$, &c. &c. I have said in the previous Article that those who die in the first year ($a-a_1$) will contribute nothing, and will get nothing; those who die in the second year (a_1-a_2) will each get what they have contributed, viz., £1; those who die in the third year will get £2 and one year's improvement on £1, that is, £1 + £ r , and so on: as thus—

$$\begin{array}{l}
 (a-a_1) \times 0 \\
 (a_1-a_2) \times 1 \\
 (a_2-a_3) \times (1+r) \\
 (a_3-a_4) \times (1+r+r^2) \\
 (a_4-a_5) \times (1+r+r^2+r^3) \\
 (a_5-a_6) \times (1+r+r^2+r^3+r^4) \\
 \text{\&c.} \qquad \qquad \text{\&c.}
 \end{array}$$

These several multiplications, being performed to the extremity of the table, will result in a number of series alternately positive and negative; but each positive series will contain just one term in excess of each negative series, as thus: the multiplication of the two perpendicular series $a + a_1 + a_2 + a_3 + a_4 + a_5$, &c., and $-a_1 - a_2 - a_3 - a_4 - a_5$, &c., into the next perpendicular series, viz., 0 1 1 1, &c., will result in

$$\begin{array}{l} a_1 + a_2 + a_3 + a_4 + a_5, \text{ \&c.} \\ -a_2 - a_3 - a_4 - a_5, \text{ \&c.,} \end{array}$$

leaving a_1 in excess. The like multiplication of the same two series into the next perpendicular series, viz., $r, r, r, \text{ \&c.}$, will result in

$$\begin{array}{l} a_2r + a_3r + a_4r + a_5r, \text{ \&c.} \\ -a_3r - a_4r - a_5r, \text{ \&c.,} \end{array}$$

leaving a_2r in excess; and other similar multiplications, carried on to the end of the table, will result in leaving successively in excess $a_3r^2, a_4r^3, a_5r^4, \text{ \&c.}$; and the sum of all the terms thus left in excess will be $a_1 + a_2r + a_3r^2 + a_4r^3 + a_5r^4 + \text{ \&c.}$, as representing the total sums withdrawn on the death of all the original subscribers; and it is obvious, that if the sum of this series be divided by the number of original admittees (a), we shall obtain the average withdrawn by each, viz.—

$$\frac{a_1}{a} + \frac{a_2r}{a} + \frac{a_3r^2}{a} + \frac{a_4r^3}{a} + \text{ \&c.}$$

which is, in fact, Mr. Morgan's solution.

ART. 24. I have previously shown that the amount of an annuity, forborn at 4 per cent. compound interest, during the life of a child one year of age, is, when computed by Mr. Morgan's method, 137·79. This sum is, therefore, the average of all the *different sums* which have been or could have been actually withdrawn during the 94 years' existence of a fund begun when 8,650 children were each a twelvemonth old; and this amount is, necessarily, very different from the amount of 30·34 as computed by Mr. Davies' method, which is the *equal average* sum which could be actually withdrawn by each and all of 8,650 children on their successive deaths during the like 94 years' existence of the fund. In short, such a fund, distributed by Mr. Morgan's method, apportions *different sums* to the lives as they fall in at successive intervals, while the fund distributed by Mr. Davies' method apportions *the same sum* to each: in other words, the sum of 137·79 *could not by any possibility have been paid out* to each life as it fell in, without weakening the fund and prematurely exhausting it before the extinction of all the lives; while the sum of 30·34 *could have been so paid out*, and the fund would have duly maintained itself, and have become exhausted only with the last remaining life.

ART. 25. It is thus manifest that the sole difference between the two methods of solution resides in the question, whether the

average should be taken of *amounts* or of *present values*? And the reason why the two methods coincide, in the two extreme cases exhibited in Articles 12 and 16, is, firstly, because present values may be truly accumulated into amounts, when an average is not required to be taken—that is to say, when the annuity is for a term certain, and no life involved in the question; and, secondly, because *present values* are, in point of fact, identical with amounts, when no interest is allowed.

ART. 26. It will therefore appear, as I conjectured in Art. 3, that Mr. Morgan and Mr. Davies have been in reality investigating two wholly different questions. Mr. Morgan's method is what it purports to be, and is correctly described and correctly solved. Mr. Davies, on the other hand, has not solved the problem proposed by Mr. Morgan, but has correctly solved one of his own; which he has, however, incorrectly described. His problem should have been thus stated:—To determine what equal amount could be paid out of a common fund on the death of each individual of a given age out of a given number of subscribers, in consideration of the payment of an annuity of £1 for and during their lives.

ART. 27. It may be reasonably asked, what are the practical purposes to which either of the two problems can be applied? To this it may be answered, that Mr. Davies' problem, in another and better known form, is in course of daily and hourly application in the general business of life assurance, and in questions relating thereto. Mr. Morgan's problem, on the other hand, does not so immediately appear to be applicable to any of the questions which usually present themselves to the practical actuary, and it is difficult to imagine for what purpose that able writer could have computed the seventh table in his work; but as I have ever entertained, with Professor De Morgan, the opinion, that the labours of competent men are never absolutely wasted, however useless or inapplicable they may at first sight appear to be—but, on the contrary, that they will in due time bear valuable fruit—so I fully believe that some practical application of his singular problem may hereafter be discovered. I think it not altogether unreasonable to conjecture that some question may have presented itself to his acute and practical mind, as to what would have been the probable effects on the funds of the Equitable of a forbearance to distribute, as they accrued, some of the large accumulations of that Society. This problem would seem to me to exactly meet the requirements of such a question.
